Penalty Method for a Vanilla Option

We apply a penalty method for an American-style vanilla option, with value function $V(S,t)$ for the price S of the underlying, strike K and maturity T. That is, we solve the Black–Scholes-type PDE

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV + p(V) = 0\,, \quad p(V) := \frac{\epsilon C}{V + \epsilon - q}\,,$$

where $q = K - S$ for a put and $q = S - K$ for a call. The nonlinear function $p(V)$ is the penalty term. The parameters $\epsilon > 0$ and $C > 0$ should be chosen such that $p \geq 0$, with $p \approx 0$ for S in the continuation area and $p > 0$ for S in the stopping area. If the penalty approach works, than its solution $V_{\epsilon,C}(S,t)$ approximates the true solution.

Example: put with $K = 10$, $T = 1$, interest rate $r = 0.25$, volatility $\sigma = 0.6$, dividend rate $\delta = 0.2$. For the parameters* we choose

$$C = rK\,, \quad \epsilon = 2C\Delta t\,, \quad \Delta t = T/M\,.$$

The shape of p in Figure 1 illustrates the functioning of the payoff term. The steep gradient indicates the location of the free boundary, and the vanishing p for larger S activates the Black–Scholes part of the PDE.

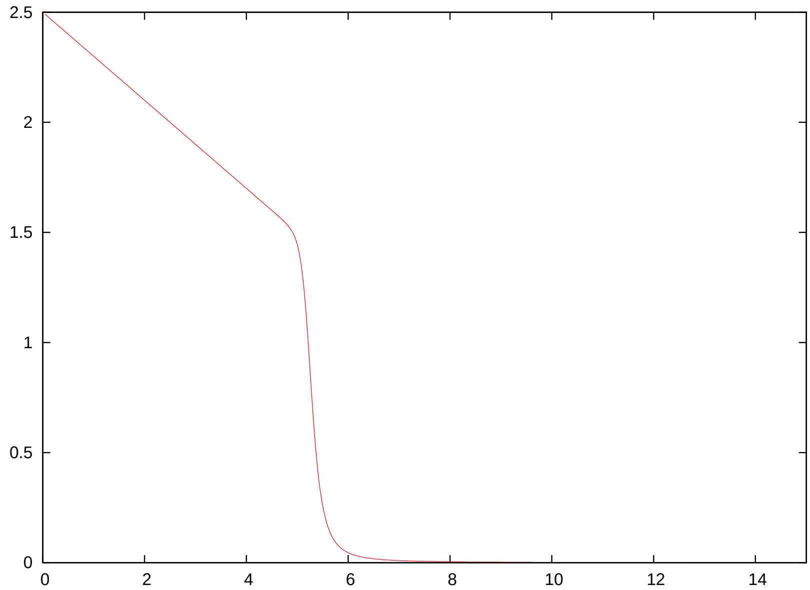

Figure 1 depicts $p(V_{\epsilon,C}(S,0))$ over S.

* For the example see also Figure 4.13 in [Tools fCF], and for the penalty term [B.F. Nielsen, O. Skavhaug, A. Tveito (2008)].

The free boundary is not defined precisely for the penalty method. Rather there is a boundary layer around the smooth-contact point. This layer is indicated in Figure 2.

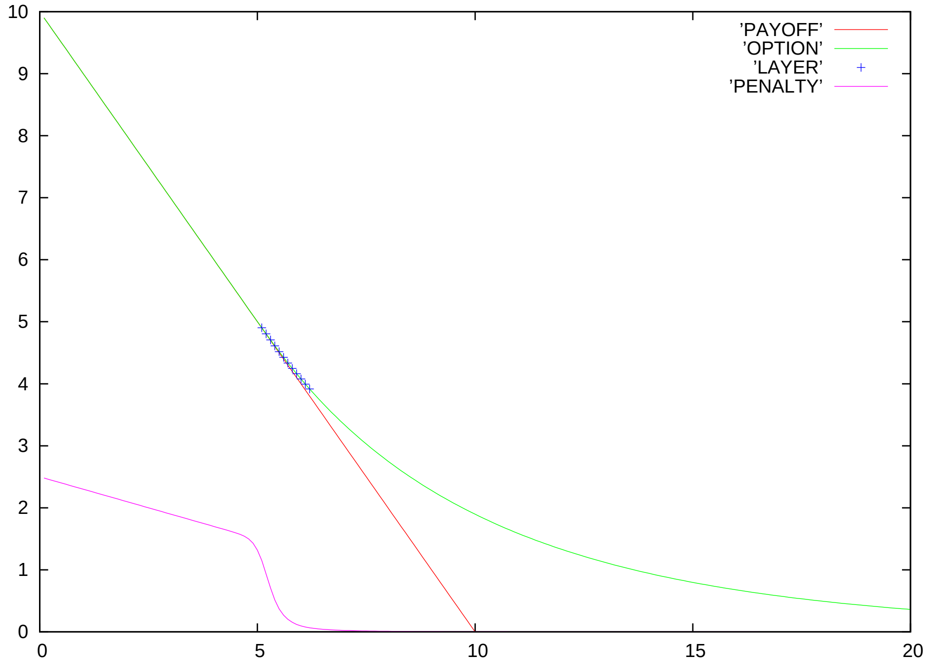

Figure 2: payoff over S, $V_{\epsilon,C}(S,0)$, penalty p, and contact layer (+)

Figure 3 shows the surface defined by $V_{\epsilon,C}(S,t)$ in green for $4 \leq S \leq 12$, $0 \leq t \leq 1$; the smooth-contact layer is indicated by red color ($M = 800$).

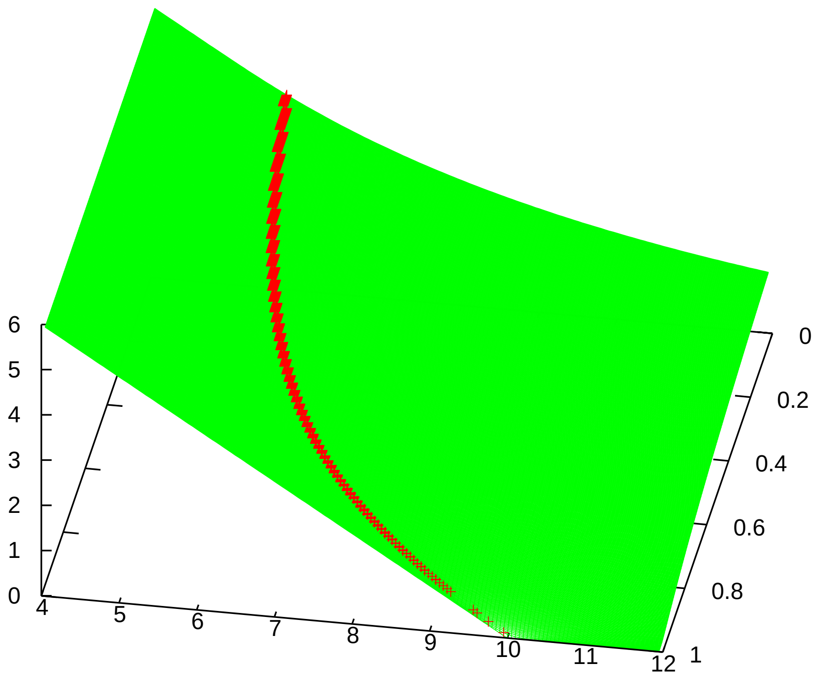

Figure 3: solution $V_{\epsilon,C}(S,t)$ (in green) with smooth-contact layer (in red)