Bifurcation, the Structure of Dynamics

A Nonmathematical Introduction

Rüdiger Seydel

based on Section1.1 of

R. Seydel's *Practical Bifurcation and Stability Analysis*

February 2024

Every day of our lives we experience changes that occur either gradually or suddenly. We often characterize these changes as quantitative or qualitative, respectively. For example, consider the following simple experiment (Figure 1). Imagine a board supported at both ends, with a load on top. If the load λ is not too large, the board will take a bent shape with a deformation depending on the magnitude of λ and on the board's material properties (such as stiffness, K). This state of the board will remain *stable* in the sense that a small variation in the load λ (or in the stiffness K) leads to a state that is only slightly perturbed. Such a variation (described by Hooke's law) would be referred to as a quantitative change. The board is deformed within its elastic regime and will return to its original shape when the perturbation in λ is removed.

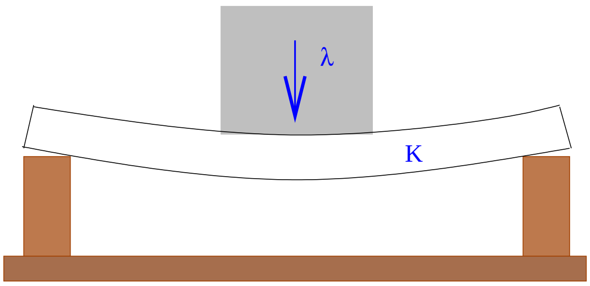

Figure 1: Bending of a board (white) with stiffness K under load λ

The situation changes abruptly when the load λ is increased beyond a certain *critical level* λ_0 at which the board breaks (Figure 2b). This sudden action is an example of a qualitative change; it will also take place when the material properties are changed beyond a certain limit (see Figure 2a). Suppose the shape of the board is modeled by some function (solution of an equation). Loosely speaking, we may say that there is a solution for load values $\lambda < \lambda_0$ and that this solution ceases to exist for $\lambda > \lambda_0$. The load λ and stiffness K are examples of *parameters*.

The outcome of any experiment, any event, and any construction is controlled by parameters. The practical problem is to *control the state* of a system—that is, to find parameters such that the state fulfills our requirements. This role of parameters is occasionally emphasized by terms such as *control parameter*, or *design parameter*. Varying a

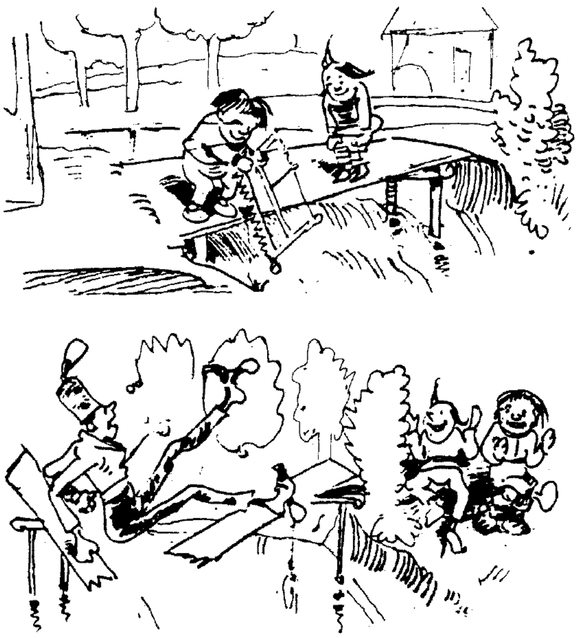

Figure 2: Manipulating K [From W. Busch. After the original hand drawing in the Wilhelm-Busch-Museum, Hannover]

parameter can result in a transition from a quantitative change to a qualitative change. The following pairs of verbs may serve as illustrations:

bend	$\rightarrow$	break
incline	$\rightarrow$	tilt over
stretch	$\rightarrow$	tear
inflate	$\rightarrow$	burst

The verbs on the left side stand for states that are stable under small perturbations; the response of each system is a quantitative one. This behavior ends abruptly at certain critical values of underlying parameters. The related drastic and irreversible change is reflected by the verbs on the right side. Close to a critical threshold the system becomes most sensitive; tiny perturbations may trigger drastic changes. To control a system may mean to find parameters such that the state of the system is safe from being close to a critical threshold. Since reaching a critical threshold often is considered as failure, the control of parameters is a central part of risk control.[1]

For example, the response of a system to variation of a parameter might look as the situation in Figure 3. We see the temporal variation of a reaction in a chemical or biological system, where the parameter drifts from the value $\lambda = 0.1$ to the value $\lambda = 0.3$ within the time interval $0 \leq t \leq 200$. The vertical axis might represent

[1] The prototypical *tilt over* (in this context already in the first edition of 1988) has become popular as name for such phenomena (in German: Kippen, or Kipppunkt). But this somewhat negative meaning is too limited to adequately describe the phantastic aspects of bifurcation. For example, Hopf bifurcation plays a pivot role in dynamics. One should rather use the neutral term *bifurcation*.

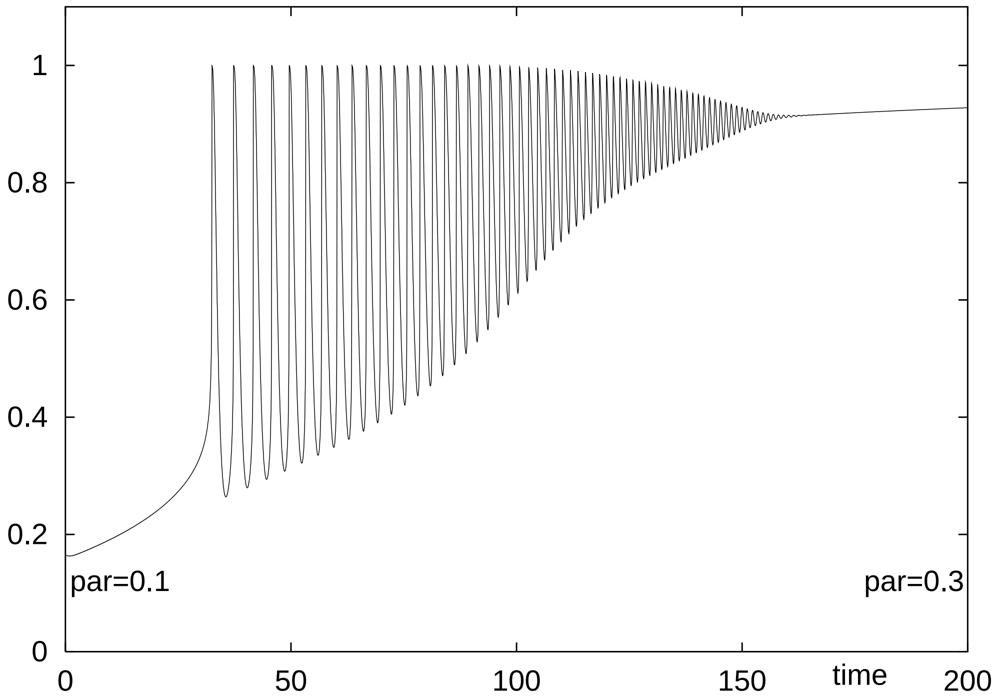

Figure 3: Response of a system to a parameter that gradually grows from 0.1 to 0.3

the temperature of a reaction, and the parameter λ could be the opening of a valve. Initially, for $t \approx 0$, the observed state variable reacts only moderately to the growing parameter. Then, all of a sudden at around time $t \approx 30$, when the parameter λ passes approximately the value 0.13, a large-amplitude oscillation sets in. The regime has changed drastically. With the parameter growing further, the oscillation slowly dies out. Finally, the state becomes again stationary ($t \approx 150$, $\lambda \approx 0.25$), and the state of the system has entered another regime. This third regime differs from the first regime by its significantly higher level.

It is interesting to note what has happened when the parameter passed the interval $0.1 \leq \lambda \leq 0.3$: Two critical threshold values were passed, and there was a "hard loss of stability" of the first regime, which goes along with a jump in the state variable. Analyzing the system under consideration closer, reveals the underlying structure, which is the *skeleton* of the dynamical behavior. This is illustrated by Figure 4, where the skeleton is built in ("bifurcation diagram"). The two horizontal axes of the parameter and of the time match. Two threshold values (in Figure 4 the "bifurcations" of the heavy line[2]) initiate the dynamical switches between qualitatively different regimes.

The above-mentioned threshold values are first examples of a class of phenomena that we denote with the term *bifurcation*. A key mechanism is indicated by the pair

$$\text{stationary state} \quad \leftrightarrow \quad \text{motion.}$$

Let us mention a few examples. The electric membrane potential of nerves is stationary as long as the stimulating current remains below a critical threshold; if this critical value

[2]example of Hopf bifurcations

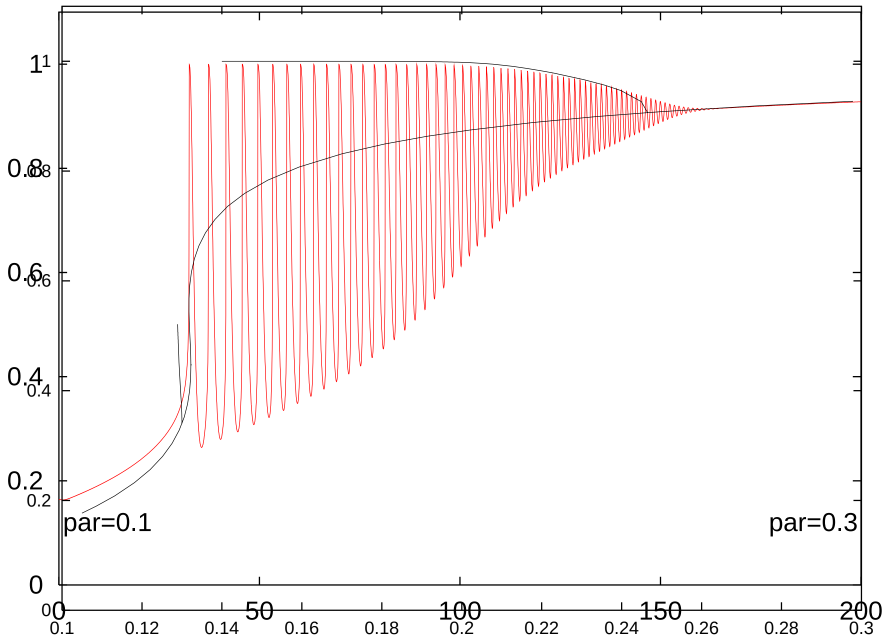

Figure 4: The skeleton of Figure 3: two plots in one figure. Here the time-dependent oscillation of Figure 3 is shown in red color (horizontal axis: time $0 \leq t \leq 200$). The underlying bifurcation diagram (in black) reveals the structure of the dynamics, characterizing the way the oscillation changes with the parameter (horizontal axis: parameter $0.1 \leq \lambda \leq 0.3$). Two bifurcations of the black curves are shown. Birth and death of the oscillation set in at the bifurcations.

is passed, the membrane potential begins to oscillate, resulting in nerve impulses. The motion of a semitrailer is straight for moderate speeds (assuming the rig is steered straight); if the speed exceeds a certain critical value, the vehicle tends to sway. Or take the fluttering of a flag, which will occur only if the moving air passes fast enough. Similarly, the vibration of tubes depends on the speed of the internal fluid flow and on the speed of an outer flow. This type of oscillation also occurs when obstacles, such as bridges and other high structures, are exposed to strong winds. Many other examples—too complex to be listed here—occur in combustion, fluid dynamics, and geophysics.

The transition from a stationary state to motion, and vice versa, is also a qualitative change. Here, speaking again in terms of *solutions*—of governing equations—we have a different quality of solution on either "side" of a critical parameter (Figures 3, 4). Let the parameter in question again be denoted by λ, with critical value λ_0. Thinking, for instance, in terms of the state variable wind speed, the state (e.g., of a flag or bridge) is stationary for $\lambda < \lambda_0$ and oscillatory for $\lambda > \lambda_0$. Qualitative changes may come in several steps, as indicated by the sequence

stationary state
regular motion
irregular motion.

The transition from regular to irregular motion is related to the onset of turbulence, or "chaos." — As a first tentative definition, we will denote a qualitative change caused by the variation of some physical (or chemical or biological, etc.) parameter λ as *bifurcation*. We will use the same symbol λ for various kinds of parameters. Some examples of parameters are listed in the Table.

TABLE. Examples of parameters.

Phenomenon	**Controlled by a typical parameter**
Bending of a rod	Load
Vibration of an engine	Frequency or imbalance
Combustion	Temperature
Nerve impulse	Generating potential
Superheating	Strength of external magnetic field
Oscillation of an airfoil	Speed of plane relative to air
Climatic changes	Solar radiation

Some important features that may change at bifurcations have already been mentioned. The following list summarizes various kinds of qualitative changes:

$$\begin{array}{rcl} \text{stable} & \leftrightarrow & \text{unstable} \\ \text{symmetric} & \leftrightarrow & \text{asymmetric} \\ \text{stationary} & \leftrightarrow & \text{periodic (regular) motion} \\ \text{regular} & \leftrightarrow & \text{irregular} \\ \text{order} & \leftrightarrow & \text{chaos} \end{array}$$

Several of these changes may take place simultaneously in complicated ways.

The quality of solutions or states is also distinguished by their geometrical shape—that is, by their *pattern*. For example, the five patterns in Figure 5 characterize five possibilities of how a state variable varies with time. The solution profile of Figure 5(a) is "flat" or stationary, the state remains at a constant level. Figure 5(b) shows a wavy pattern with a simple periodic structure. The patterns of Figure 5(c) and (d) are again wavy but less regular, and the pattern of Figure 5(e) appears to be irregular (chaotic). The five different patterns of Figure 5 arise for different values of a parameter λ; new patterns form when the parameter passes critical values. This example illustrates why such bifurcation phenomena are also called *pattern formation*. — Figure 5 shows an example of an isothermal reaction.[3] Such transitions are typical for a wide range of problems. A similar sequence of patterns is, for example, the velocity of the reaction front, where the first profile (a) stands for a uniformly propagating combustion front, and the wavy pattern (b) represents a regularly pulsating front.

So far this introduction has stressed the situation where the state of the system varies with time—that is, the focus has been on *temporal dynamics*. In addition, the state of a system may also vary with space. For example, animal coats may have spots

[3]from [A. Bayliss, B.J. Matkowsky: Two routes to chaos in condensed phase combustion. SIAM J.Appl.Math. 50 (1990) 437].

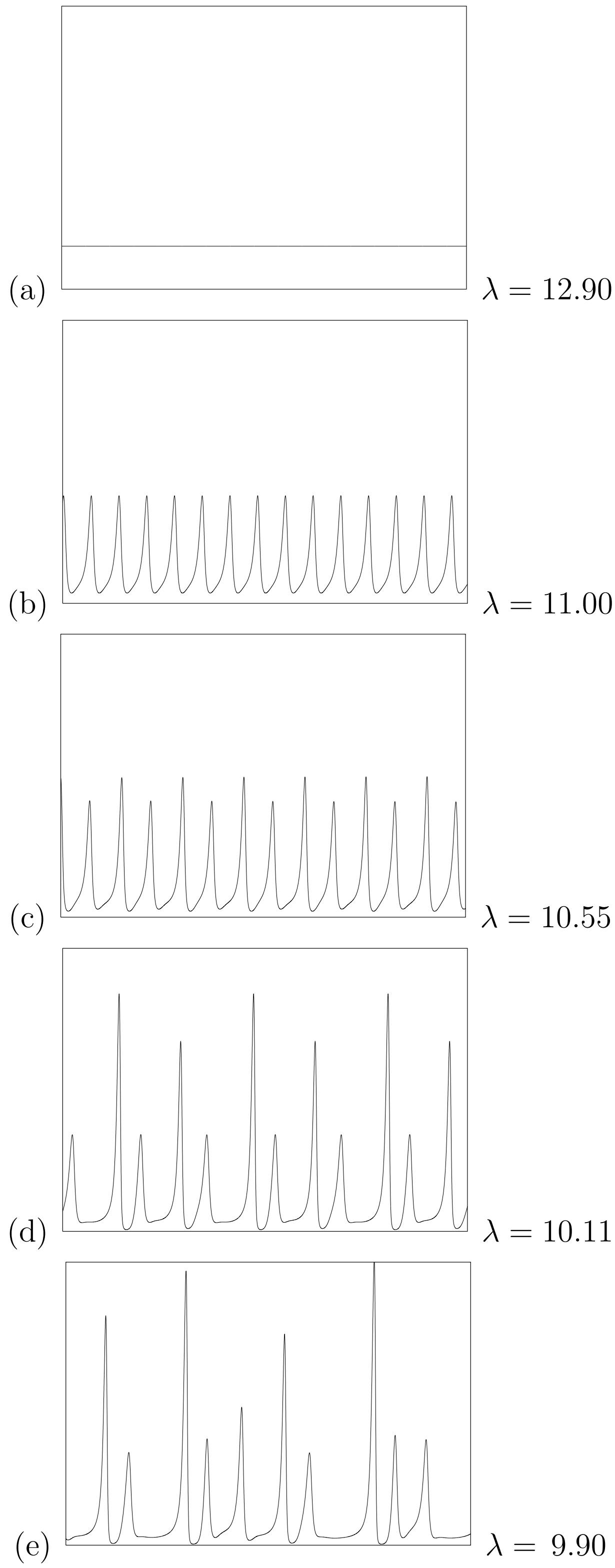

Figure 5: Changing structure or pattern: growing complexity with decreasing parameter λ; (a): stationary state, (b): periodic state, (c): periodic with double period, (d): fourfold period, (e): aperiodic motion ("chaos"). This is an example of an isothermal reaction; parameter values are shown on the right; each of the five boxes depicts time-dependent temperature $y(t)$ for $0 \leq t \leq 10$, the vertical axes are scaled such that $0 \leq y \leq 85$.

or stripes, which can be explained by variations of morphogens. If the morphogen is non-uniformly distributed (the *heterogeneous* state) a pattern of spots or stripes develops. No pattern develops in case the morphogens are distributed homogeneously. The pair

$$\text{homogeneous} \quad \leftrightarrow \quad \text{heterogeneous}$$

is the spatial analog to the pair "stationary $\leftrightarrow$ motion" that stresses temporal dynamics. Problems in full generality will often display both temporal and spatial dynamics. For example, a chemical reaction may show a concentration with spiral-wave pattern that migrates slowly across the disk.

Transitions among different patterns (as depicted in Figure 5) are ubiquitous. For instance, cardiac rhythm is described by similar patterns. One of the possible patterns may be more desirable than others. Hence, one faces the problem of how to switch patterns. By means of a proper external stimulus one can try to give the system a "kick" such that it hopefully changes its pattern to a more favorable state. For example, heart beat can be influenced by electrical stimuli. The difficulties are to decide how small a stimulus to choose, and how to set the best time instant for stimulation. One pattern may be more robust and harder to disturb than another pattern that may be highly sensitive and easy to excite. Before manipulating the transition among patterns, mechanisms of *pattern selection* must be studied. Which *structure* is most *attractive*? Which states are stable? For which values of the parameters is the system most sensitive?

To obtain related knowledge, a thorough discussion of bifurcation phenomena is necessary, which requires language, tools and insight of mathematics.

Reference:

R. Seydel: Practical Bifurcation and Stability Analysis. Interdisciplinary Applied Mathematics Vol. 5, Third Edition. Springer, New York 2010

(First Edition: From Equilibrium to Chaos. Practical Bifurcation and Stability Analysis. Elsevier 1988)