
Preface

Preface to the Sixth Edition

Five years after the fifth edition came out, there is a need to include additional results, and to improve explanations of methods and algorithms further. Besides numerous enhancements of small details there are new subjects.

New in the sixth edition are, for example, methods for smoothing in Chapter 1, and an introduction into basic aspects of efficiency. In Chapter 2, acceptance-rejection methods for generating random numbers are explained, with application to the Ziggurat algorithm for calculating normal variates. Chapter 3 on Monte Carlo methods now includes a subsection on positive solutions, and an outline of the antithetic variance reduction. Iterative approaches in Chapter 4 have become less important, in favor of direct methods.

To support the important role tree methods play in practice, an entire new appendix (Appendix D) is devoted these methods. This appendix includes trinomial methods, multidimensional trees, and implied trees for variable volatility. Further, how to handle discrete dividends is explained.

And the text is enriched by more figures. To facilitate understanding, many of the figures have been recalculated to become colored. Additional formalized exercises are included, and numerous hints at informal exercises are spread throughout the text.

On the technical side, the entire book has been transferred from *plain-T_EX* to *L_AT_EX*. This has offered plenty of occasion to work the book over thoroughly. Additional colored figures can be found in the collection *Topics for Computational Finance* (shortly *Topics fCF*) on the internet platform www.compfin.de.

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Preface to the Fifth Edition

Financial engineering and numerical computation are genuinely different disciplines. But in finance many computational methods are used and have become indispensable. This book explains how computational methods work in financial engineering. The main focus is on computational methods; financial engineering is the application. In this context, the numerical methods are *tools*, the tools for computational finance.

Faced with the vast and rapidly growing field of financial engineering, we need to choose a subarea to avoid overloading the textbook. We choose the attractive field of option pricing, a core task of financial engineering and risk analysis. The broad field of option pricing is both ambitious and diverse enough to call for a wide range of computational tools. Confining ourselves to option pricing enables a more coherent textbook and avoids being distracted away from computational issues. We trust that the focus on option-related methods is representative of, or least helpful for, the entire field of computational finance.

The book starts with an introductory Chapter 1, which collects financial and stochastic background. The remaining parts of the book are devoted to computational methods. Organizing computational methods, roughly speaking, leads to distinguish stochastic and deterministic approaches. By “stochastic methods” we mean computations based on random numbers, such as Monte Carlo simulation. Chapters 2 and 3 are devoted to such methods. In contrast, “deterministic methods” are frequently based on solving partial differential equations. This is discussed in Chapters 4, 5 and 6. In the computer, finally, everything is deterministic. The distinction between “stochastic” and “deterministic” is mainly to motivate and derive different approaches.

All of the computational methods must be adapted to the underlying model of a financial market. Here we meet different kinds of stochastic processes, from geometric Brownian motion to Lévy processes. Based on the chosen process an option model is selected. The classical choice is the Black–Scholes model for vanilla options with one underlying asset. This benchmark market model is “complete” in that all claims can be replicated. Established by Black, Merton, Scholes and others, this model is the main application of methods explained in Chapters 2 – 6. Chapter 7 goes beyond and addresses more general models. Allowing for jump processes, transaction costs, multi-asset underlyings, or more complicated payoffs, leads to incomplete markets. Computational methods for incomplete markets are briefly discussed in Chapter 7.

This book has been published in several editions. The first German edition (2000) was mainly absorbed by the Black–Scholes equation. Later editions (first English edition 2002) were carefully opened to more general models and a wider selection of methods. The book has grown with the development of the field. Faced with a large variety of possible computational tools, this book attempts to balance the need for a sufficient number of powerful algorithms

with the limitations of a textbook. The balance has been gradually shifting over the years and editions. Numerous investigations in our research group have influenced the choice of covered topics. We have implemented and tested many dozens of algorithms, and gained insight and experience. A significant part of this knowledge has entered the book.

Readership

This book is written from the perspective of an applied mathematician. The level of mathematics is tailored to advanced undergraduate science and engineering majors. Apart from this basic knowledge the book is self-contained and can be used for a course on the subject. The intended readership is interdisciplinary and includes professionals in financial engineering, mathematicians and scientists of many other fields.

An expository style may attract a readership ranging from students to practitioners. Methods are introduced as tools for immediate application. Formulated and summarized as algorithms, a straightforward implementation in computer programs should be possible. In this way, the reader may learn by computational experiment. *Learning by calculating* will be a possible way to explore several aspects of the financial world. In some parts, this book provides an algorithmic introduction to computational finance. To keep the text readable for a wide audience, some aspects of proofs and derivations are exported to exercises at which hints are frequently given.

New in the Fifth Edition

The revisions to this fifth edition are much more extensive than those of previous editions. Compared to the fourth edition, the page count has increased by about 100 pages. The main addition is Chapter 7, which is devoted to incomplete markets. It begins with an introduction to nonlinear Black–Scholes type partial differential equations, as they arise from considering transaction costs or ranges for a stochastic volatility. Numerical approaches require instruments that converge to viscosity solutions. These solutions are introduced in an appendix. The role of monotonicity of numerical schemes is outlined. Lévy processes, with a focus on Merton’s jump-diffusion and a numerical approach to the resulting partial integro-differential equation are then addressed. The chapter ends with an exposition on how the Fourier transform can be applied to option pricing. To complete the introduction of more general models and methods, the Dupire equation is outlined in a new appendix.

In addition to the new Chapter 7, several larger extensions and new Sections have been written for this edition. The calculation of Greeks is described in more detail, including the method of adjoints for a sensitivity analysis (new Section 3.7). Penalty methods are introduced and applied to a two-factor model in the new Section 6.7. More material is presented in the field of analytical methods; in particular, Kim’s integral representation and its computation have been added to Chapter 4. Tentative guidelines on how to compare different algorithms and judge efficiency are given in the new Section 4.9. The

chapter on finite elements has been extended with a discussion of two-asset options.

Apart from additional material listed above, the entire book has been thoroughly revised. The clarity of the expository parts has been improved; all sections have been tested in the class room. Numerous amendments, further figures, exercises and many references have been added. For example, the principal component analysis and its applications are included and the role of different boundary conditions is outlined in more detail.

How to Use this Textbook

Exercises are stated at the end of each chapter. They range from easy routine tasks to laborious projects. In addition to these explicitly formulated exercises, plenty of “hidden” exercises are spread throughout the book, with comments such as “the reader may check.” Of course, the reader is encouraged to fill in those small intermediate steps that are excluded from the text.

This book explains the basic ideas of several approaches, presenting more material than is accomplishable in one semester. The following guidelines have proved successful in teaching:

First Course:

Chapter 1 without Section 1.6.2,
Chapter 2,
Chapter 3 without Section 3.7,
Chapter 4, with one analytic method out of Section 4.8,
and without Section 4.9,
Chapter 6, or parts of it.

Second Course:

the remaining parts, in particular
Chapter 5 and Chapter 7.

Depending on the detail of explanation, the first course could be for undergraduate students. The second course may attract graduate students.

Extensions in the Internet

There is an accompanying internet page:

www.compfin.de

This is intended to serve the needs of the computational finance community and provides complementary material to this book. In particular, the collection *Topics in Computational Finance*, which is under construction, presents several of our findings or figures that would go beyond the limited scope of a textbook. In its final state, *Topics* is anticipated as a companion volume to the *Tools*.

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Contents

1	Modeling Tools for Financial Options	1
1.1	Options	1
1.1.1	The Payoff Function	3
1.1.2	A Priori Bounds	5
1.1.3	Options in the Market	6
1.1.4	The Geometry of Options	7
1.2	Model of the Financial Market	9
1.3	Numerical Methods	12
1.3.1	Algorithms	13
1.3.2	Discretization	14
1.3.3	Efficiency	15
1.4	The Binomial Method	16
1.4.1	A Discrete Model	16
1.4.2	Derivation of Equations	17
1.4.3	Solution of the Equations	20
1.4.4	A Basic Algorithm	21
1.4.5	Improving the Convergence	23
1.4.6	Sensitivities	27
1.4.7	Extensions	28
1.5	Risk-Neutral Valuation	28
1.6	Stochastic Processes	32
1.6.1	Wiener Process	33
1.6.2	Stochastic Integral	36
1.7	Diffusion Models	39
1.7.1	Itô Process	39
1.7.2	Geometric Brownian Motion	41
1.7.3	Risk-Neutral Valuation	43
1.7.4	Mean Reversion	45
1.7.5	Vector-Valued Stochastic Differential Equations	46
1.8	Itô Lemma and Applications	48
1.8.1	Itô Lemma	49

1.8.2	Consequences for Geometric Brownian Motion	50
1.8.3	Integral Representation	51
1.8.4	Bermudan Options	52
1.8.5	Empirical Tests	54
1.9	Jump Models	55
1.9.1	Poisson Process	56
1.9.2	Simulating Jumps	57
1.9.3	Jump Diffusion	58
1.10	Calibration	60
1.11	Notes and Comments	63
1.12	Exercises	68
2	Generating Random Numbers with Specified Distributions	79
2.1	Uniform Deviates	80
2.1.1	Linear Congruential Generators	80
2.1.2	Quality of Generators	81
2.1.3	Random Vectors and Lattice Structure	81
2.1.4	Fibonacci Generators	85
2.2	Extending to Random Variables from Other Distributions	86
2.2.1	Inversion	87
2.2.2	Transformation in $\mathbb{R}^1$	89
2.2.3	Transformations in $\mathbb{R}^n$	91
2.2.4	Acceptance-Rejection Method	91
2.3	Normally Distributed Random Variables	92
2.3.1	Method of Box and Muller	92
2.3.2	Variant of Marsaglia	94
2.3.3	Ziggurat	95
2.3.4	Correlated Random Variables	98
2.4	Monte Carlo Integration	99
2.5	Sequences of Numbers with Low Discrepancy	102
2.5.1	Discrepancy	102
2.5.2	Examples of Low-Discrepancy Sequences	105
2.6	Notes and Comments	108
2.7	Exercises	110
3	Monte Carlo Simulation with Stochastic Differential Equations	117
3.1	Approximation Error	119
3.2	Stochastic Taylor Expansion	122
3.3	Examples of Numerical Methods	125
3.3.1	Positivity	126
3.3.2	Runge-Kutta Methods	127
3.3.3	Taylor Scheme with Weak Second-Order Convergence	127
3.3.4	Higher-Dimensional Cases	129
3.4	Intermediate Values	129

3.5	Monte Carlo Simulation	131
3.5.1	Integral Representation	131
3.5.2	Basic Version for European Options	132
3.5.3	Accuracy	136
3.5.4	Variance Reduction	139
3.5.5	Application to an Exotic Option	143
3.5.6	Test with Halton Points	145
3.6	Monte Carlo Methods for American Options	145
3.6.1	Stopping Time	146
3.6.2	Parametric Methods	148
3.6.3	Regression Methods	151
3.7	Sensitivity	155
3.7.1	Pathwise Sensitivities	156
3.7.2	Adjoint Method	158
3.8	Notes and Comments	160
3.9	Exercises	162
4	Standard Methods for Standard Options	169
4.1	Preparations	170
4.2	Foundations of Finite-Difference Methods	172
4.2.1	Difference Approximation	172
4.2.2	The Grid	173
4.2.3	Explicit Method	175
4.2.4	Stability	177
4.2.5	An Implicit Method	180
4.3	Crank–Nicolson Method	180
4.4	Boundary Conditions	184
4.5	Early-Exercise Structure	187
4.5.1	Early-Exercise Curve	188
4.5.2	Free-Boundary Problem	190
4.5.3	Black–Scholes Inequality	193
4.5.4	Penalty Formulation	194
4.5.5	Obstacle Problem	195
4.5.6	Linear Complementarity for American Put Options	197
4.6	Computation of American Options	198
4.6.1	Discretization with Finite Differences	199
4.6.2	Reformulation and Analysis of the LCP	201
4.6.3	Iterative Procedure for the LCP	202
4.6.4	Direct Method for the LCP	203
4.6.5	An Algorithm for Calculating American Options	205
4.7	On the Accuracy	208
4.7.1	Elementary Error Control	209
4.7.2	Extrapolation	213
4.8	Analytic Methods	213
4.8.1	Approximation Based on Interpolation	215

4.8.2	Quadratic Approximation	218
4.8.3	Analytic Method of Lines	220
4.8.4	Integral-Equation Method	223
4.8.5	Other Methods	225
4.9	Criteria for Comparisons	226
4.10	Notes and Comments	229
4.11	Exercises	235
5	Finite-Element Methods	243
5.1	Weighted Residuals	244
5.1.1	The Principle of Weighted Residuals	246
5.1.2	Examples of Weighting Functions	249
5.1.3	Examples of Basis Functions	249
5.1.4	Smoothness	250
5.2	Ritz–Galerkin Method with One-Dimensional Hat Functions ..	251
5.2.1	Hat Functions	251
5.2.2	Assembling	253
5.2.3	A Simple Application	255
5.3	Application to Standard Options	257
5.3.1	European Options	257
5.3.2	Variational Form of the Obstacle Problem	259
5.3.3	Variational Form of an American Option	260
5.3.4	Implementation of Finite Elements	261
5.4	Two-Asset Options	265
5.4.1	Analytical Preparations	265
5.4.2	Weighted Residuals	266
5.4.3	Boundary	268
5.4.4	Involved Matrices	270
5.5	Error Estimates	273
5.5.1	Strong and Weak Solutions	274
5.5.2	Approximation on Finite-Dimensional Subspaces	276
5.5.3	Quadratic Convergence	277
5.6	Notes and Comments	281
5.7	Exercises	282
6	Pricing of Exotic Options	289
6.1	Exotic Options	290
6.2	Options Depending on Several Assets	292
6.3	Asian Options	296
6.3.1	The Payoff	296
6.3.2	Modeling in the Black–Scholes Framework	297
6.3.3	Reduction to a One-Dimensional Equation	298
6.3.4	Discrete Monitoring	300
6.4	Numerical Aspects	304
6.4.1	Convection-Diffusion Problems	305

6.4.2	Von Neumann Stability Analysis	307
6.5	Upwind Schemes and Other Methods	309
6.5.1	Upwind Scheme	309
6.5.2	Dispersion	312
6.6	High-Resolution Methods	314
6.6.1	Lax–Wendroff Method	314
6.6.2	Total Variation Diminishing	315
6.6.3	Numerical Dissipation	316
6.7	Penalty Method for American Options	318
6.7.1	LCP Formulation	318
6.7.2	Penalty Formulation	319
6.7.3	Discretization of the Two-Factor Model	320
6.8	Notes and Comments	323
6.9	Exercises	325
7	Beyond Black and Scholes	329
7.1	Nonlinearities in Models for Financial Options	330
7.1.1	Leland’s Model of Transaction Costs	330
7.1.2	The Barles and Soner Model of Transaction Costs	332
7.1.3	Specifying a Range of Volatility	333
7.1.4	Market Illiquidity	336
7.2	Numerical Solution of Nonlinear Black–Scholes Equations	336
7.2.1	Transformation	337
7.2.2	Discretization	339
7.2.3	Convergence of the Discrete Equations	340
7.3	Option Valuation under Jump Processes	344
7.3.1	Characteristic Functions	345
7.3.2	Option Valuation with PIDEs	348
7.3.3	Transformation of the PIDE	349
7.3.4	Numerical Approximation	350
7.4	Application of the Fourier Transform	353
7.5	Notes and Comments	357
7.6	Exercises	359
A	Financial Derivatives	363
A.1	Investment and Risk	363
A.2	Financial Derivatives	364
A.3	Forwards and the No-Arbitrage Principle	366
A.4	The Black–Scholes Equation	368
A.5	Early-Exercise Curve	373
A.6	Equations with Volatility Function	376

B Stochastic Tools	381
B.1 Essentials of Stochastics	381
B.2 More Advanced Topics	386
B.3 State-Price Process	388
C Numerical Methods	393
C.1 Basic Numerical Tools	393
C.2 Iterative Methods for $Ax = b$	400
C.3 Function Spaces	403
C.4 Minimization	405
C.5 Viscosity Solutions	408
D Extended Tree Methods	411
D.1 Convergence to the Black–Scholes Formula	411
D.2 Discrete Dividend Payment	413
D.3 Trinomial Trees	418
D.4 Multidimensional Trees	420
D.5 Implied Trees for Variable Volatility	423
E Complementary Material	427
E.1 Bounds for Options	427
E.2 Approximation Formula	429
E.3 Software	432
References	433
Index	451